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Superpositions of Continuous Functions

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In this note we present relatively short and simple proofs for some theorems concerning superpositions of functions. We prove in particular the well known theorem of Kolmogorov [8], and its generalization due to Ostrand [10].

Our main observation is that by combining a general duality argument of functional analysis with the ideas introduced by Lorentz [9] and Hedberg [5], we obtain a better understanding of the nature of these theorems, and can avoid some of the difficulties which arose in former proofs.

We use the notation of [3]. $C(X)$ is the Banach space of real valued continuous functions on the compact metric space X , with the norm $\|f\| = \sup_{x \in X} |f(x)|$. We identify the dual $C(X)^*$ of $C(X)$ with the space of real regular Borel measures on X with the total variation as norm. μ^+ (resp. μ^-) denotes the positive (resp. negative) part of the real measure μ , and $|\mu| = \mu^+ + \mu^-$. Clearly

$$\|\mu\| = \| |\mu| \| = \mu^+(X) + \mu^-(X). \quad (1)$$

If φ is a continuous function which maps X onto some (compact metric) space Y , and $\mu \in C(X)^*$, then $\mu \circ \varphi$ is the element of $C(Y)^*$ defined by

$$\mu \circ \varphi(V) = \mu(\varphi^{-1}(V)), \quad V \subset Y. \quad (2)$$

We denote the interval $[0, 1]$ by I , the n dimensional cube by I^n , and the circle by T . $\dim X$ is the covering dimension of X .

DEFINITION 1. Let X be a compact metric space. Let F be a family of continuous functions on X . We say that F *uniformly separates the Borel measures on X* if there exists a constant λ , $0 < \lambda \leq 1$, such that for each $\mu \in C(X)^*$, $\|\mu \circ \varphi\| \geq \lambda \|\mu\|$ for some $\varphi \in F$.

Let us say a word about the intuitive meaning of this concept: if F uniformly separates the Borel measures on X , and H_1, H_2 are disjoint closed subsets of X , then for some $\varphi \in F$ the intersection $\varphi[H_1] \cap \varphi[H_2]$ is "not too large,"

where "not too large" depends on λ , and on a measure μ in $C(X)^*$ such that H_1, H_2 are the supports of μ^+, μ^- respectively. In particular a family of functions which uniformly separates Borel measures, separates points. (Given $x_1 \neq x_2$ in X , apply the definition to $\mu = \delta_{x_1} - \delta_{x_2}$.) The converse is false: let $X = I^2$, let $F = \{\varphi_1, \varphi_2\}$ where $\varphi_1(x, y) = x$ and $\varphi_2(x, y) = y$. Clearly F separates points, but for $\mu = \delta_{(0,0)} + \delta_{(1,1)} - \delta_{(0,1)} - \delta_{(1,0)}$ we have $\|\mu\| = 4$, and $\|\mu \circ \varphi_i\| = 0, i = 1, 2$. i.e. F does not uniformly separate Borel measures. See [12] where this concept as well as related topics are studied.

The connection between uniform separation and superpositions is given in the following.

THEOREM 1. *Let $F = \{\varphi_i\}_{i=1}^k$ be a finite family of continuous functions on a compact metric space X , with $\varphi_i[X] = Y_i, 1 \leq i \leq k$. The family F uniformly separates the Borel measures on X if and only if each $f \in C(X)$ can be represented as*

$$f(x) = \sum_{i=1}^k g_i(\varphi_i(x)) \quad (3)$$

with $g_i \in C(Y_i), 1 \leq i \leq k$.

Proof. Let Y denote the disjoint union of the Y_i 's, $1 \leq i \leq k$. Consider the bounded linear operator $S: C(Y) \rightarrow C(X)$ defined by

$$Sg(x) = \sum_{i=1}^k g(\varphi_i(x)), \quad g \in C(Y), \quad x \in X. \quad (4)$$

A routine check shows that the adjoint S^* of S acts according to the formula

$$S^*\mu = \sum_{i=1}^k \mu \circ \varphi_i, \quad \mu \in C(X)^*, \quad (5)$$

and that

$$\|S^*\mu\| = \sum_{i=1}^k \|\mu \circ \varphi_i\|. \quad (6)$$

Each $f \in C(X)$ admits a representation of the form (3) if and only if S maps $C(Y)$ onto $C(X)$. This occurs if and only if S^* is an *isomorphism into*, i.e., there exists a constant $\alpha > 0$ such that $\|S^*\mu\| \geq \alpha \|\mu\|$ for all $\mu \in C(X)^*$ (see [3]).

By 6 this is equivalent to F being uniformly separating Borel measures on X . ■

A very simple illustration of an application of Theorem 1 is

THEOREM 2. *There exists three real valued analytic functions $\{\varphi_i\}_{i=1}^3$ on the circle T , such that each $f \in C(T)$ can be represented as $f(t) = \sum_{i=1}^3 g_i(\varphi_i(t))$ with $g_i \in C(I)$.*

(See Kahane [6] for a similar result. The number three in Theorem 2 cannot be reduced as proved in [11].)

Proof. We realize T as the interval I with its endpoints identified. Set

$$I_1 = (0, \tfrac{1}{3}), \quad I_2 = (\tfrac{1}{3}, \tfrac{2}{3}), \quad I_3 = (\tfrac{2}{3}, 1) \quad (\text{Open intervals}) \quad (7)$$

and

$$J_i = T \setminus I_i, \quad i = 1, 2, 3. \quad (8)$$

Let φ_i , $i = 1, 2, 3$ be any three elements of $C(T)$ such that $\varphi_i|_{J_i}$ is one to one. ($\varphi_i|_{J_i}$ is the restriction of φ_i to J_i ; the same notation will be used later for measures.) We claim that $F = \{\varphi_i\}_{i=1}^3$ uniformly separates the Borel measures on T , with $\lambda = \frac{1}{3}$.

Indeed, let $\mu \in C(T)^*$ be of norm one. Then $|\mu|$ is a probability measure, and it is easily seen that

$$\sum_{i=1}^3 |\mu|(J_i) = \sum_{i=1}^3 \int 1_{J_i} d|\mu| = \int \left(\sum_{i=1}^3 1_{J_i} \right) d|\mu| \geq 2 \quad (9)$$

since $\sum_{i=1}^3 1_{J_i}(t) \geq 2$ for all $t \in T$. (1_{J_i} is the indicator function of J_i .) It follows that $|\mu|(J_{i_0}) \geq \frac{2}{3}$ for some i_0 , $1 \leq i_0 \leq 3$. Thus, $\|\mu \circ (\varphi_{i_0}/J_{i_0})\| \geq \frac{2}{3}$ since φ_{i_0} is one to one on J_{i_0} .

Clearly $|\mu|(I_{i_0}) \leq \frac{1}{3}$, hence, the mass of μ which is outside J_{i_0} , can reduce the norm of $\mu \circ (\varphi_{i_0}/J_{i_0})$ by at most $\frac{1}{3}$, i.e.,

$$\|\mu \circ \varphi_{i_0}\| \geq \|\mu \circ (\varphi_{i_0}/J_{i_0})\| - |\mu|(I_{i_0}) \geq \frac{2}{3} - \frac{1}{3} = \frac{1}{3}. \quad (10)$$

Thus F uniformly separates the Borel measures on T with $\lambda = \frac{1}{3}$, and the theorem follows from Theorem 1. ■

The proofs of the theorems of Kolmogorov and Ostrand require more machinery. We start with some more definitions.

DEFINITION 2. (a) A family U of subsets of a metric space X is said to be *discrete* if its elements have mutually disjoint closures.

(b) $\delta(U)$ is $\sup_{\mathcal{U} \in U}$ diameter \mathcal{U} .

(c) If φ is a function on X , we say that φ *separates* U if for each $\mathcal{U}_1, \mathcal{U}_2 \in U$, $\varphi[\mathcal{U}_1] \cap \varphi[\mathcal{U}_2] = \emptyset$.

(d) If U_1, U_2, \dots, U_k are k families of subsets of X we say that $\{U_i\}_{i=1}^k$ covers the set X n times ($n \leq k$) if each $x \in X$ is an element in some member of U_i for at least n values of i .

The following are trivial observations. (We do not distinguish between U_i and the union of its elements.)

PROPOSITION 1. Let X be a set, and let $\{U_i\}_{i=1}^k$ be k families of subsets of X . The statements (a), (b), (c), (d) are equivalent and imply (e).

(a) $\{U_i\}_{i=1}^k$ covers X n times.

(b) Each $k - n + 1$ of the families $\{U_i\}_{i=1}^k$ cover X .

(c) $\sum_{i=1}^k 1_{U_i}(x) \geq n$ for all $x \in X$.

(d) $\sum_{i=1}^k \mu(U_i) \geq n$ for all probability measures μ on X .

(e) For each probability measure μ on X there exists some i_0 , $0 \leq i_0 \leq k$, so that $\mu(U_{i_0}) \geq n/k$.

LEMMA 1. Let X be a compact metric space, and let $F = \{\varphi_i\}_{i=1}^k$ be a family of continuous functions on X . If for each $\epsilon > 0$, there exists k finite discrete families U_1, U_2, \dots, U_k of subsets of X so that

(i) $\{U_i\}_{i=1}^k$ covers X $\left[\frac{k}{2}\right] + 1$ times,

(ii) $\delta(U_i) < \epsilon$, $1 \leq i \leq k$,

(iii) φ_i separates U_i , $1 \leq i \leq k$.

Then F uniformly separates the Borel measures on X with $\lambda = 1/k$.

Proof. We wish to show that for each $\mu \in C(X)^*$, $\|\mu \circ \varphi_i\| \geq (1/k) \|\mu\|$ for some $\varphi_i \in F$. The measures μ with μ^+ and μ^- having disjoint supports are norm dense in $C(X)^*$, (by regularity) and therefore we may consider such measures only.

So let $\mu \in C(X)^*$ be of norm one, and with $\text{supp } \mu^+ \cap \text{supp } \mu^- = \emptyset$. Let $\epsilon = d(\text{supp } \mu^+, \text{supp } \mu^-)$, and let $\{U_i\}_{i=1}^k$ be the families of sets corresponding to ϵ .

It follows that a member of U_i , $1 \leq i \leq k$ cannot intersect both $\text{supp } \mu^+$ and $\text{supp } \mu^-$.

By (i) and Proposition 1(e), there exists $1 \leq i_0 \leq k$ so that

$$|\mu|(U_{i_0}) \geq \frac{1}{k} \left(\left[\frac{k}{2} \right] + 1 \right) \geq \frac{1}{k} \left(\frac{k}{2} + \frac{1}{2} \right) = \frac{1}{2} + \frac{1}{2k}. \quad (11)$$

Now, since members of U_{i_0} intersect at most one of the sets $\text{supp } \mu^+$ and $\text{supp } \mu^-$, and since by (iii), φ_{i_0} separates U_{i_0} , it follows from (11) that

$$\|\mu \circ (\varphi_{i_0}/U_{i_0})\| \geq \frac{1}{2} + \frac{1}{2k}. \quad (12)$$

Clearly by (11)

$$|\mu|(X \setminus U_{i_0}) \leq \frac{1}{2} - \frac{1}{2k}. \quad (13)$$

Hence, by the same reasoning as in the proof of Theorem 2, we get from (12) and (13) that

$$\begin{aligned} \|\mu \circ \varphi_{i_0}\| &\geq \|\mu \circ (\varphi_{i_0}/U_{i_0})\| - |\mu|(X \setminus U_{i_0}) \\ &\geq \frac{1}{2} + \frac{1}{2k} - \left(\frac{1}{2} - \frac{1}{2k}\right) = \frac{1}{k}. \end{aligned} \quad (14)$$

i.e., F uniformly separates the Borel measures on X with $\lambda = 1/k$. ■

By Proposition 1, condition (i) of Lemma 1 is equivalent to the following: any $[(k+1)/2]$ of the families U_i cover X . Such a cover by $[(k+1)/2]$ families U_i is of order $[k/2]$ (i.e., at most $[k/2] + 1$ of its elements intersect). It follows that the existence of families $\{U_{ij}\}_{i=1}^k$ with (i) and (ii) of Lemma 1 for each $\epsilon > 0$ implies that $\dim X \leq [k/2]$. Ostrand [10] proved the following converse assertion.

THEOREM 3. *Let X be an n -dimensional compact metric space, let $k \geq n+1$, and $\epsilon > 0$.*

There exist k discrete families $\{U_i\}_{i=1}^k$ of subsets of X which cover X $k-n$ times, so that $\delta(U_i) < \epsilon$, $1 \leq i \leq k$.

Our next lemma proves the existence of functions $\{\varphi_i\}_{i=1}^k$ as in the assumption of Lemma 1, if we are given a suitable sequence of nice coverings of X . We shall do this in a more general setting which will be used later on.

LEMMA 2. *Let X_j , $j = 1, 2, \dots, L$ be compact metric spaces and let $X = X_1 \times X_2 \times \dots \times X_L$.*

For each $1 \leq j \leq L$ let $\{U_m^j\}_{m=1}^\infty$ be a sequence of discrete families of subsets of X_j with $\delta(U_m^j) \rightarrow_{m \rightarrow \infty} 0$.

Let U_m , $m = 1, 2, \dots$ be the family of subsets of X defined by

$$U_m = \{\mathcal{U}^1 \times \mathcal{U}^2 \times \dots \times \mathcal{U}^L; \mathcal{U}^j \in U_m^j\}. \quad (15)$$

and let $\lambda_1, \lambda_2, \dots, \lambda_L$ be reals independent over the rationals. There exist functions $\tau_j \in C(X_j)$, $1 \leq j \leq L$ such that the function $\varphi \in C(X)$ defined by

$$\varphi(x_1, x_2, \dots, x_L) = \sum_{j=1}^L \lambda_j \tau_j(x_j)$$

separates U_m for infinitely many m 's.

If $X_1 = X_2 = \dots = X_L$ and $U_m^1 = U_m^2 = \dots = U_m^L$ then one can also take $\tau_1 = \tau_2 = \dots = \tau_L$.

Proof. Set $C = C(X_1) \times C(X_2) \times \dots \times C(X_L)$ with the norm $\|(\tau_1, \tau_2, \dots, \tau_L)\| = \max_{1 \leq j \leq L} \|\tau_j\|$.

For each integer $\ell \geq 1$ let $A_\ell \subset C$ be defined by

$$A_\ell = \left\{ (\tau_1, \tau_2, \dots, \tau_L) : \varphi(x_1, x_2, \dots, x_L) = \sum_{j=1}^L \lambda_j \tau_j(x_j) \right. \\ \left. \text{separates } U_m \text{ for some } m \geq \ell \right\}. \quad (16)$$

We claim that A_ℓ is open and dense in C for all $\ell \geq 1$.

A_ℓ is open: Let $\tau = (\tau_1, \tau_2, \dots, \tau_L) \in A_\ell$, i.e. $\varphi = \sum_{j=1}^L \lambda_j \tau_j(x_j)$ separates U_m for some $m \geq \ell$.

$\epsilon = \inf_{\mathcal{U}, \mathcal{T} \in U_m} d(\varphi[\mathcal{U}], \varphi[\mathcal{T}])$ is positive since U_m is discrete. Let $\delta > 0$ be so small that $\|\tau - \tau'\|_C < \delta$ implies $\|\varphi - \varphi'\|_{C(X)} < \epsilon/2$ where $\varphi'(x_1, x_2, \dots, x_L) = \sum_{j=1}^L \lambda_j \tau'_j(x_j) \in C(X)$. Then φ' separates U_m too, i.e., $\tau' \in A_\ell$.

A_ℓ is dense: Let $\psi = (\psi_1, \psi_2, \dots, \psi_L) \in C$ and $\epsilon > 0$ be given.

We shall construct $\tau \in A_\ell$ with $\|\tau - \psi\| < \epsilon$.

Let $m \geq \ell$ be so big that the oscillation of ψ_j on elements of U_m^j is smaller than ϵ for all $1 \leq j \leq L$. Such an m exists since $\delta(U_m^j) \rightarrow_{m \rightarrow \infty} 0$.

Let τ_j be defined as follows: τ_j is constant on elements of U_m^j , these constants being *distinct* rationals so that $\|\tau_j/U_m^j - \psi_j/U_m^j\| < \epsilon$. This being possible by the above choice of m , we extend τ_j to the whole of X_j by Tietze's theorem so that $\|\tau_j - \psi_j\| < \epsilon$. Then clearly $\|\tau - \psi\| < \epsilon$ where $\tau = (\tau_1, \tau_2, \dots, \tau_L)$. We claim that $\tau \in A_\ell$. Indeed, let $\mathcal{U} = \mathcal{U}^1 \times \mathcal{U}^2 \times \dots \times \mathcal{U}^L \in U_m$, with $\mathcal{U}^j \in U_m^j$ and $\tau_j[\mathcal{U}^j] = r_j$ —the rational value of τ_j on the element \mathcal{U}^j of U_m^j .

If $\varphi(x_1, x_2, \dots, x_L) = \sum_{j=1}^L \lambda_j \tau_j(x_j)$, then φ attains the constant value $\sum_{j=1}^L \lambda_j r_j$ on \mathcal{U} . But all the reals $\sum_{j=1}^L \lambda_j r_j$ are distinct, since the λ_j 's are independent over the rationals, and the values of τ_j on members of U_m^j are distinct rationals. It follows that φ separates U_m , i.e., $\tau \in A_\ell$.

Let $A = \bigcap_{\ell=1}^{\infty} A_\ell$. By the Baire category theorem A is a dense G_δ in C , and each $\tau = (\tau_1, \tau_2, \dots, \tau_L) \in A$ has the desired property.

If $X_1 = X_2 = \dots = X_L$ and $U_m^1 = U_m^2 = \dots = U_m^L$ the same arguments can be applied with the sets $A_\ell \subset C(X_1)$, $A_\ell = \{\tau \in C(X_1): \varphi(x_1, x_2, \dots, x_L) = \sum_{j=1}^L \lambda_j \tau(x_j) \text{ separates } U_m \text{ for some } m \geq \ell\}$. This proves Lemma 2. ■

Remark. If $X_1 = X_2 = \dots = X_L = I$, (i.e., $X = I^n$) and the elements of $U_m^1 = U_m^2 = \dots = U_m^L$ are intervals then one can extend the τ_j 's from U_m^j to I by letting them being linear on the intervals in the complement of U_m^j , provided the length of these complementing intervals tends to 0 together with the intervals in U_m^j . (This will be the case in our proof of Kolmogorov's theorem.)

Moreover: $C(X_j) = C(I)$ can be replaced in this case by $\text{Lip}_\alpha(I)$, $0 < \alpha < 1$, i.e., the τ_j 's can be chosen to be (nondecreasing) $\text{Lip } \alpha$ functions. (See [5].)

KOLMOGOROV'S THEOREM. Let $n \geq 2$. There exist functions ψ_i , $i = 1, 2, \dots, 2n + 1$ in $C(I)$ and reals $\lambda_1, \lambda_2, \dots, \lambda_n$ such that each $f \in C(I^n)$ can be represented as

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{2n+1} g_i \left(\sum_{j=1}^n \lambda_j \psi_i(x_j) \right), \quad g_i \in C(R).$$

Proof. Set $k = 2n + 1$. For each m , consider a partition of I into m intervals of length $1/m$ each, indexed from 1 to m by the natural order (i.e., the first is $[0, 1/m]$ and the last $[(m-1)/m, 1]$). Let $V_{m,i}$, $1 \leq i \leq k$ be the family of intervals generated by removing from I those intervals of the above partition with index congruent to $i \pmod k$. (All intervals in $V_{m,i}$, except the two extreme ones, are of length $(k-1)/m$, and for each m , $\{V_{m,i}\}_{i=1}^k$ covers I $k-1$ times.)

Set $U_{m,i} = \{I_1 \times I_2 \times \dots \times I_n: I_j \in V_{m,i}\}$, $1 \leq i \leq k$. It is easy to check that $\{U_{m,i}\}_{i=1}^k$ covers I^n , $k-n = [k/2] + 1$ times.

Let $\{\lambda_j\}_{j=1}^n$ independent over the rationals (e.g., $\lambda_j = e^{j-1}$). By Lemma 2 there exists functions ψ_i in $C(I)$, $1 \leq i \leq k$ and a subsequence $\{m_r\}_{r=1}^\infty$ of the integers such that $\varphi_i(x_1, x_2, \dots, x_n) = \sum_{j=1}^n \lambda_j \psi_i(x_j) \in C(I^n)$ separates $U_{m_r,i}$ for $r = 1, 2, \dots$. By Lemma 1, $\{\varphi_i\}_{i=1}^k$ uniformly separates the Borel measures on I^n , and the theorem follows from Theorem 1. ■

OSTRAND'S THEOREM. Let $X = X_1 \times X_2 \times \dots \times X_L$ where X_j is a compact metric space of dimension n_j , $1 \leq j \leq L$. Let $n = \sum_{j=1}^L n_j$.

I. There exists functions $\{\psi_{i,j}\}_{i=1}^{2n+1}$, $1 \leq j \leq L$ in $C(X_j)$ such that each $f \in C(X)$ can be represented as

$$f(x_1, x_2, \dots, x_L) = \sum_{i=1}^{2n+1} g_i \left(\sum_{j=1}^L \psi_{i,j}(x_j) \right) \quad \text{with} \quad g_i \in C(R).$$

II. If $X_1 = X_2 = \dots = X_L$, then one can take $\psi_{i,j} = \lambda_j \psi_i$ where $\{\lambda_j\}_{j=1}^L$ are reals independent over the rationals.

Proof. Set $k = 2n + 1$. For each $m \geq 1$ let $U_{m,i}^j$ be a discrete family of subsets of X_j , $1 \leq i \leq k$, so that

- (a) $\{U_{m,i}^j\}_{i=1}^k$ covers X_j , $k - n_j$ times, for $1 \leq j \leq L$.
- (b) $\delta(U_{m,i}^j) \rightarrow_{m \rightarrow \infty} 0$ for $1 \leq j \leq L$, and $1 \leq i \leq k$.

Such families exist by Theorem 3. Set

$$U_{m,i} = \{\mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_L: \mathcal{U}_j \in U_{m,i}^j\}, \quad 1 \leq i \leq k, \quad m = 1, 2, \dots$$

From (a) and (b) it follows that

- (i) $\{U_{m,i}\}_{i=1}^k$ cover X $k - n = [k/2] + 1$ times for each $m = 1, 2, \dots$,
- (ii) $\delta(U_{m,i}) \rightarrow_{m \rightarrow \infty} 0$ for $1 \leq i \leq k$.

By Lemma 2 there exists a subsequence $\{m_r\}_{r=1}^\infty$ of the integers, and functions $\{\psi_{i,j}\}_{i=1}^k$ in $C(X_j)$ so that $\varphi_i(x_1, x_2, \dots, x_L) = \sum_{j=1}^L \psi_{i,j}(x_j) \in C(X)$ separates $U_{m_r,i}$ for all $r = 1, 2, \dots$ and $1 \leq i \leq k$. (We apply Lemma 2 for $i = 1$ first to get $\{\tau_{1,j}\}_{j=1}^L$ and set $\psi_{1,j} = \lambda_j \tau_{1,j}$. $\varphi_1(x_1, x_2, \dots, x_L) = \sum_{j=1}^L \psi_{1,j}(x_j)$ separates $U_{m_r,1}$ for infinitely many m_r 's, and we can apply Lemma 2 again with $i = 2$ on this subsequence to get $\{\psi_{2,j}\}_{j=1}^L$ and so on.)

By Lemma 1 $\{\varphi_i\}_{i=1}^k$ separates the Borel measures on X , and the theorem follows from Theorem 1. For II just apply the second part of Lemma 2. ■

Remarks. The number $2n + 1$ in both Kolmogorov's and Ostrand's theorems cannot be reduced, at least not for $n = 2, 3, 4$ (see [11] and [12]).

As remarked after the proof of Lemma 2, the functions ψ_i in Kolmogorov's theorem can be chosen in $\text{Lip}_\alpha(I)$, $\alpha < 1$. Fridman [4] proved that the ψ_i 's can even be Lip 1 functions. (See also Kahane [14] for a short proof.) However, the ψ_i 's cannot be chosen to be continuously differentiable, as proved by Vituskin and Henkin [13], and Kaufman [7].

Demko [1] recently extended Kolmogorov's theorem to bounded continuous functions on R^n , while Doss [2] proved that addition can be replaced by multiplication in this theorem.

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